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Note

An improved product construction for large sets of Kirkman triple systems☆

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Abstract

It has been shown by Lei, in his recent paper, that there exists a large set of Kirkman triple systems of order uv (LKTS(uv)) if there exist an LKTS(v), a TKTS(v) and an LR(u), where a TKTS(v) is a transitive Kirkman triple system of order v, and an LR(u) is a new kind of design introduced by Lei. In this paper, we improve this product construction by removing the condition "there exists a TKTS(v)". Our main idea is to use transitive resolvable idempotent symmetric quasigroups instead of TKTS. As an application, we can combine the known results on LKTS and LR-designs to obtain the existence of an LKTS($3^n m(2\cdot 13^{n_1}+1)\cdots(2\cdot 13^{n_t}+1)$) for $n\geqslant 1$, $m\in\{1,5,11,17,25,35,43,67,91,123\}\cup\{2^{2r+1}25^s+1: r\geqslant 0, s\geqslant 0\}$, $t\geqslant 0$ and $n_i\geqslant 1$ $(i=1,\ldots,t)$.

Keywords: LR-design; Large set of Kirkman triple systems; Transitive resolvable idempotent symmetric quasigroup

1. Introduction

This paper is a further work of [14]. We first recall some definitions.

A *group-divisible design* (briefly GDD) is a triple $(X,\mathcal{G},\mathcal{B})$ which satisfies the following properties: (i) X is a finite set of points; (ii) $\mathcal{G}$ is a partition of X into subsets called groups; (iii) $\mathcal{B}$ is a set of subsets of X (called blocks) such that a group and a

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block contain at most one common point, and any pair of points from distinct groups occur in exactly one block.

A GDD $(X,\mathscr{G},\mathscr{B})$ is called *resolvable* if there exists a partition $\Gamma=\{P_1,P_2,\dots,P_r\}$ of $\mathscr{B}$ such that each part P_i (called *parallel class*) is a partition of X.

A GDD is called *transversal design* if it has exactly u groups of size t and every block has size u. We denote such a GDD by $\mathrm{TD}(u,t)$. A transversal design is called *resolvable* if it is resolvable as a GDD.

A GDD $(X,\mathscr{G},\mathscr{B})$ is called *Steiner triple system* if $|X|=v$ and it has v groups of size 1 and every block has size 3. Such a GDD is denoted briefly by $\mathrm{STS}(v)$ $(X,\mathscr{B})$. A resolvable $\mathrm{STS}(v)$ is called *Kirkman triple system* and denoted $\mathrm{KTS}(v)$. Note that there are exactly $v(v-1)/6$ blocks in a $\mathrm{KTS}(v)$, all blocks are partitioned into $(v-1)/2$ parallel classes. A $\mathrm{KTS}(v)$ is known to exist whenever $v\equiv 3\ (\mathrm{mod}\ 6)$, see [9]. A $\mathrm{KTS}(v)$ $(X,\mathscr{B})$ is called *transitive*, and denoted $\mathrm{TKTS}(v)$, if there exists a group G of order v acting transitively on X, which forms an automorphism group of $(X,\mathscr{B})$.

A large set of $\mathrm{KTS}(v)$ (denoted $\mathrm{LKTS}(v)$) is a collection of $v-2$ pairwise disjoint $\mathrm{KTS}(v)$ on the same point set. Note that the necessary condition for the existence of $\mathrm{LKTS}(v)$ is $v\equiv 3\ (\mathrm{mod}\ 6)$.

Here we summarize the content of [14].

A quasigroup is a pair $(X,\circ)$, where X is a set and $(\circ)$ is a binary operation on X such that equations $a\circ x=b$ and $y\circ a=b$ are uniquely solvable for every pair of elements a,b in X. The order of a quasigroup $(X,\circ)$ is the size of X. A quasigroup of order v is called *idempotent* and *symmetric* if identities $x^2=x$ and $x\circ y=y\circ x$ hold for all x, y in X. An idempotent symmetric quasigroup of order v will be denoted by $\mathrm{ISQ}(v)$. Furthermore, an $\mathrm{ISQ}(v)$ $(X,\circ)$ is called (*sharply*) *transitive*, if there exists a group G of order v acting transitively on X which forms an automorphism group of $(X,\circ)$.

Suppose that $(X,\circ)$ is an $\mathrm{ISQ}(v)$. Any given pair of elements x and y in X determines uniquely a product $x\circ y$ and then a triad $(x,y,x\circ y)$. Note that since $(X,\circ)$ is symmetric, we can view the two triads $(x,y,x\circ y)$ and $(y,x,y\circ x)$ as the same. A quasigroup $(X,\circ)$ is called *resolvable* if all $\binom{v}{2}$ pairs of distinct elements can be partitioned into subsets T_i $(1\leqslant i\leqslant 3(v-1)/2)$, such that every $\Gamma_i=\{(x,y,x\circ y)\colon\ \{x,y\}\in T_i\}$ (called parallel class) forms a partition of X. A transitive resolvable $\mathrm{ISQ}(v)$ is denoted by $\mathrm{TRISQ}(v)$.

The existence of TRISQ is known.

Theorem 1.1 (Zhang and Zhu [14, Theorem 2.3]). *A* $\mathrm{TRISQ}(v)$ *exists if and only if* $v\equiv 3\ (\mathrm{mod}\ 6)$.

TRISQ has been used to modify the well known Denniston's tripling construction for LKTS [5].

Theorem 1.2 (Zhang and Zhu [14, Lemma 3.1]). *If there exist a* $\mathrm{TRISQ}(v)$ *and an* $\mathrm{LKTS}(v)$, *then there exists an* $\mathrm{LKTS}(3v)$.

The main result of [14] is as follows.

Theorem 1.3 (Zhang and Zhu [14, Theorem 3.2]). *If there exists an* LKTS(v), *then there exists an* LKTS($3v$).

Recently, Lei [7] introduced a new concept called LR-design. He presented a product construction to obtain an LKTS(uv) from an LKTS(v), a TKTS(v) and an LR(u). In Section 2, we modify this construction in the same spirit of [14] by using TRISQ instead of TKTS. Hence, the above condition "there exists a TKTS(v)" can be removed. In Section 3, we apply the improved product construction to obtain LKTS with new orders. The best known existence of LKTS is then stated in Theorem 3.3.

2. An improved construction for LKTS

We first recall the definition of LR-design [7].

Let X be a set of size u. An *LR-design* of order u (briefly LR(u)) is a collection $\{(X,\mathscr{A}_k^j); 1\leqslant k\leqslant (u-1)/2,\ j=0,1\}$ of $u-1$ KTS(u)s with following properties:

(i) Let the resolution of $\mathscr{A}_k^j$ be $\Gamma_k^j=\{A_k^j(h); 1\leqslant h\leqslant (u-1)/2\}$. There is an element in each Γ_k^j, which without loss of generality, we can suppose is $A_k^j(1)$, such that

$$\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1)=\bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1)=\mathscr{A}$$

and $(X,\mathscr{A})$ is a KTS(u).

(ii) For any triple $T=\{x,y,z\}\subset X$, $x\neq y\neq z\neq x$, there exist k,j such that $T\in\mathscr{A}_k^j$.

The following recursive construction for LKTS is presented in [7].

Theorem 2.1. *If there exist both an* LKTS(v) *and a* TKTS(v), *and there exists an* LR(u), *then there exists an* LKTS(uv).

Now we modify the proof of the above theorem, by using TRISQ instead of TKTS, to obtain the following result.

Theorem 2.2. *If there exist both an* LKTS(v) *and an* LR(u), *then there exists an* LKTS(uv).

Proof. Suppose that X_1 is a set of size u with a linear order "$<$" (i.e. for any $x\neq y$, $x,y\in X$, either $x<y$ or $y<x$). We have an LR(u) over X_1 with the following collection of $u-1$ KTS(u)

$$\left\{(X_1,\mathscr{A}_k^j):\ 1\leqslant k\leqslant\frac{u-1}{2},\ j=0,1\right\}$$

with following properties:

(i) Let the resolution of $\mathscr{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h):\ 1 \leqslant h \leqslant (u-1)/2\}$, and

$$\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1) = \mathscr{A},$$

$(X_1, \mathscr{A})$ is a KTS(u).

(ii) For any triple $T = \{x, y, z\} \subset X_1$, $x \neq y \neq z \neq x$, there exist k, j such that $T \in \mathscr{A}_k^j$.

Furthermore, suppose that X_2 is a set of size v and $(X_2, \mathscr{B}_j)$ $(1 \leqslant j \leqslant v-2)$ is an LKTS(v). Assume that $\mathscr{B}_j$ is partitioned into parallel classes $\{B_j(m):\ 1 \leqslant m \leqslant (v-1)/2\}$. Since $v \equiv 3 \pmod 6$, we have a TRISQ(v) over X_2 by Theorem 1.1. Let $(X_2, \circ)$ be a TRISQ(v). We will construct an LKTS(uv) on the point set $X_1 \times X_2$. The construction proceeds in 3 steps.

Step 1: Let $G = \{\sigma_0, \sigma_1, \ldots, \sigma_{v-1}\}$ be the transitive automorphism group of $(X_2, \circ)$ written multiplicatively. For any $\{x, y, z\} \subseteq X_1$, $x < y < z$, $\sigma_i, \sigma_j \in G$ and $a \in X_2$, define

$$B_{ija}^{(x,y,z)} = \{(x, a), (y, \sigma_j(a)), (z, \sigma_i \sigma_j^2(a))\},$$
$$P_{ij}^{(x,y,z)} = \{B_{ija}^{(x,y,z)}:\ a \in X_2\}$$

and

$$\mathscr{A}_i^{(x,y,z)} = \bigcup_{\sigma_j \in G} P_{ij}^{(x,y,z)}.$$

Then each $(\{x, y, z\} \times X_2, \mathscr{A}_i^{(x,y,z)})$ $(i = 0, 1, \ldots, v-1)$ is a resolvable TD$(3, v)$ with the parallel classes $P_{ij}^{(x,y,z)}$, $\sigma_j \in G$, and these v TDs form a large set of disjoint resolvable TDs.

Now suppose $\{(x, y, z)\} \in \mathscr{A}$, $x < y < z$. For each $x' \in X_1$, we have $v-2$ disjoint KTS(v) $(\{x'\} \times X_2, \mathscr{B}_j^{(x')})$ for $j = 1, 2, \ldots, v-2$, where $\mathscr{B}_j^{(x')} = \{x'\} \times \mathscr{B}_j = \{\{(x', a), (x', b), (x', c)\}:\ \{a, b, c\} \in \mathscr{B}_j\}$. For $j = 1, 2, \ldots, v-2$, define

$$\mathscr{C}_j = \left(\bigcup_{\{x,y,z\} \in \mathscr{A}} \mathscr{A}_j^{(x,y,z)} \right) \cup \left(\bigcup_{x' \in X_1} \mathscr{B}_j^{(x')} \right).$$

It is not difficult to check that each $(X_1 \times X_2, \mathscr{C}_j)$ is a KTS(uv) with the following parallel classes:

$$C_j(m) = \bigcup_{x' \in X_1} (\{x'\} \times B_j(m)), \quad 1 \leqslant m \leqslant \frac{v-1}{2},$$
$$C_j(k, i) = \bigcup_{\{x,y,z\} \in A_k^0(1)} P_{ij}^{(x,y,z)}, \quad 1 \leqslant k \leqslant \frac{u-1}{2},\ \ 0 \leqslant i \leqslant v-1.$$

Furthermore, these $v-2$ KTSs are obviously disjoint.

(The remaining resolvable TDs $\mathscr{A}_0^{(x,y,z)}$ and $\mathscr{A}_{v-1}^{(x,y,z)}$, for each $\{x,y,z\}\in\mathscr{A}$, are saved for the use in the following two steps.)

Step 2: We will make use of $\mathscr{A}_0^{(x,y,z)}$ for $\{x,y,z\}\in\mathscr{A}$. Note that $\mathscr{A}_0^{(x,y,z)}$ has v parallel classes $P_{0j}^{(x,y,z)}$.

For a given $k, 1\leqslant k\leqslant(u-1)/2$ and $\sigma_j\in G, 1\leqslant j\leqslant v-1$, take $\{x,y,z\}\in A_k^0(1)\subseteq\mathscr{A}$, $x<y<z$. Since $(X_2,\circ)$ is a TRISQ(v), for any pair $\{a,b\}\subset X_2$ $(a\neq b)$, we get an element $a\circ b$ in X_2. Furthermore, $\{a,b,a\circ b\}$ is uniquely defined by any two elements in $\{a,b,a\circ b\}$. Define

$$B_{abxj}^{(0)}=\{(x,a),(x,b),(y,\sigma_j(a\circ b))\},$$
$$B_{abyj}^{(0)}=\{(y,\sigma_j(a)),(y,\sigma_j(b)),(z,\sigma_0\sigma_j^2(a\circ b))\},$$
$$B_{abzj}^{(0)}=\{(z,\sigma_0\sigma_j^2(a)),(z,\sigma_0\sigma_j^2(b)),(x,a\circ b)\}.$$

(Note that the blocks $\{(x,a),(y,\sigma_j(a)),(z,\sigma_0\sigma_j^2(a))\}$, $\{(x,b),(y,\sigma_j(b)),(z,\sigma_0\sigma_j^2(b))\}$ and $\{(x,a\circ b),(y,\sigma_j(a\circ b)),(z,\sigma_0\sigma_j^2(a\circ b))\}$ are the three blocks of P_{0j}.)

Let

$$\mathscr{D}_{kj}^{(0)}=\left(\bigcup_{\{x,y,z\}\in A_k^0(1)}\left(P_{0j}^{(x,y,z)}\bigcup_{\substack{\{a,b\}\subset X_2\\ a\neq b}}\{B_{abxj}^{(0)},B_{abyj}^{(0)},B_{abzj}^{(0)}\}\right)\right)\bigcup_{\{x,y,z\}\in\mathscr{A}_k^0\setminus A_k^0(1)}\mathscr{A}_j^{(x,y,z)}.$$

It is not difficult to check that each $(X_1\times X_2,\mathscr{D}_{kj}^{(0)})$ is an KTS(uv). Furthermore, these $(u-1)v/2$ KTSs are disjoint.

(*Note*: Again we only give the parallel classes which consist of the following two parts:

Part I: Since $(X_2,\circ)$ is a TRISQ(v), by the definition of TRISQ, all $\binom{v}{2}$ pairs of distinct elements can be partitioned into subsets T_i $(1\leqslant i\leqslant 3(v-1)/2)$, such that every $\Gamma_i=\{(x,y,x\circ y)\colon \{x,y\}\in T_i\}$ is a partition of X_2. Then for each i $(1\leqslant i\leqslant 3(v-1)/2)$,

$$\bigcup_{\{x,y,z\}\in A_k^0(1)}\ \bigcup_{\{a,b\}\in T_i}\{B_{abxj}^{(0)},B_{abyj}^{(0)},B_{abzj}^{(0)}\}$$

is a partition of $X_1\times X_2$. Furthermore, $\bigcup_{\{x,y,z\}\in A_k^0(1)}P_{oj}^{(x,y,z)}$ is also a partition of $X_1\times X_2$. We have $(3(v-1)/2)+1$ parallel classes in this part.

Part II: For $2\leqslant m\leqslant(u-1)/2$ and $0\leqslant s\leqslant v-1$,

$$\bigcup_{\{x,y,z\}\in A_k^0(m)}P_{js}^{(x,y,z)}$$

is a partition of $X_1\times X_2$. We get $(u-3)v/2$ parallel classes.

The union of all the total $(uv-1)/2$ parallel classes is a partition of $\mathscr{D}_{kj}^{(0)}$.)

Step 3: This step is similar to Step 2. We will make use of $\mathscr{A}_{v-1}^{(x,y,z)}$ for $\{x,y,z\}\in\mathscr{A}$. Again we notice that $\mathscr{A}_{v-1}^{(x,y,z)}$ has v parallel classes $P_{v-1,j}^{(x,y,z)}$.

For a given $k, 1\leqslant k\leqslant(u-1)/2$ and $\sigma_j\in G$, take $\{x,y,z\}\in A_k^1(1)\subseteq\mathscr{A}$, $x<y<z$. Since $(X_2,\circ)$ is a TRISQ(v), for any pair $\{a,b\}\subset X_2$ $(a\neq b)$. Define

$$B_{abxj}^{(v-1)}=\{(x,a),(x,b),(z,\sigma_{v-1}\sigma_j^2(a\circ b))\},$$
$$B_{abyj}^{(v-1)}=\{(y,\sigma_j(a)),(y,\sigma_j(b)),(x,a\circ b)\},$$
$$B_{abzj}^{(v-1)}=\{(z,\sigma_{v-1}\sigma_j^2(a)),(z,\sigma_{v-1}\sigma_j^2(b)),(y,\sigma_j(a\circ b))\}$$

and

$$\mathscr{D}_{kj}^{(v-1)}=\left(\bigcup_{\{x,y,z\}\in A_k^1(1)}\left(P_{v-1,j}^{(x,y,z)}\bigcup_{\substack{\{a,b\}\subset X_2\\ a\neq b}}\{B_{abxj}^{(v-1)},B_{abyj}^{(v-1)},B_{abzj}^{(v-1)}\}\right)\right)$$
$$\bigcup_{\{x,y,z\}\in\mathscr{A}_k^1\backslash A_k^1(1)}\mathscr{A}_j^{(x,y,z)}.$$

Again each $(X_1\times X_2,\mathscr{D}_{kj}^{(v-1)})$ is an KTS(uv), and these $(u-1)v/2$ KTSs are also disjoint.

We obtain a total of $uv-2$ disjoint KTS(uv), a large set. This completes the proof of the theorem. □

3. New orders for LKTS

In this section, we will apply the above construction with the known results on LKTS to obtain new orders for LKTS.

For existence of LR-designs, we quote a theorem of Lei [7]:

Theorem 3.1. *There exists an* LR$(2\cdot 13^n+1)$ *for* $n\geqslant 1$.

With the combined efforts of many authors, see [1–6,8,12–14], some LKTS(v)s are known to exist. We summarize as follows.

Theorem 3.2. *There exists an* LKTS$(3^n m)$ *for* $n\geqslant 1$ *and* $m\in M=\{1,5,11,17,25,35,43,67,91,123\}\cup\{2^{2r+1}25^s+1:\ r\geqslant 0, s\geqslant 0\}$.

Applying Theorem 2.2 recursively with the LKTS(v)s from Theorem 3.2 and the LR(u)s from Theorem 3.1 we have the following best known existence result for LKTS(v)s. Note that the orders when $t=0$ are the same as those in Theorem 3.2.

Theorem 3.3. *There exists an* LKTS$(3^n m(2\cdot 13^{n_1}+1)(2\cdot 13^{n_2}+1)\cdots(2\cdot 13^{n_t}+1))$ *for* $n\geqslant 1$, $m\in M$, $t\geqslant 0$ *and* $n_i\geqslant 1$ $(i=1,2,\ldots,t)$.

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